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A GENERAL FORMULA FOR CALCULATING FORCES ON A 2-D ARBITRARY BODY IN INCOMPRESSIBLE FLOW

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In the present paper, a general integral equation is presented to calculate the forces exerted on a two-dimensional (2-D) body of arbitrary shape immersed in unsteady, incompressible flows. By finding the general solutions of a set of Laplace equations with particular boundary conditions, the equation can be simplified to produce a simplified formula for calculating the forces. The simplified formula consists of three parts, representing contributions from different physical phenomena: added mass force and/or inertial force in inviscid flow, the force caused by the deformation of fluid and viscosity and the force caused by the convection of fluid with nonzero circulation. It can be applied to any 2-D arbitrary body in viscous or inviscid, steady or unsteady incompressible flow. As the formula excludes either temporal derivatives of velocity or spatial derivatives of vorticity in the flow field, the numerical errors contained in the numerical solution of velocity and vorticity fields will not be magnified, and therefore the resulting force calculated is more accurate. Most importantly, the formula presents an alternative method for obtaining the added mass of a 2-D body of arbitrary shape accelerating in a fluid. For bodies of simple shape, such as a circle, ellipse and plate, the added masses predicted using the present method are in agreement with that obtained by conventional methods. For bodies of complex shape, the present method only requires the calculation of the first two coefficients of the conformal transformation and cross-sectional area.

1. INTRODUCTION

The ability to predict the forces exerted on a body immersed in an arbitrary flow field has been a major objective in fluid dynamic research. The most conventional and direct method of obtaining the forces is to integrate the elemental contributions of the pressure and the viscous shear stress over the surface of the body after the flow field has been solved numerically or analytically.

For 2-D incompressible flows, the Navier–Stokes (N–S) equations in vorticity/stream-function formulation is often solved by vortex methods because it has the advantage of only computing two unknowns (vorticity and stream function). As the numerical results obtained by vortex methods exclude the pressure field, other methods of calculating the pressure exerted on the body have to be used. There are usually the following three methods for obtaining the surface pressure.

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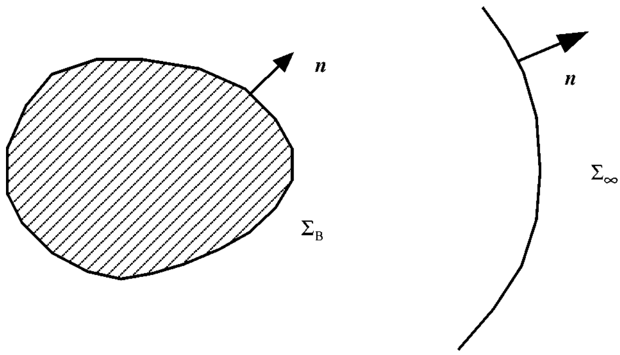


Figure 1. A two-dimensional body immersed in an incompressible fluid.

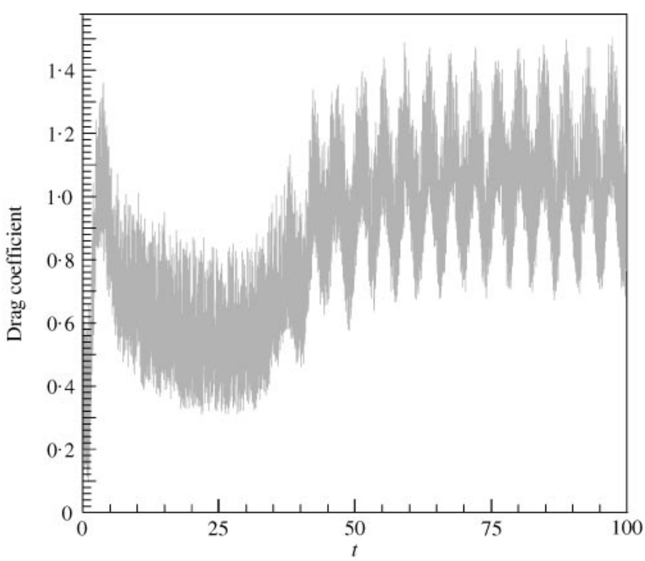


Figure 2. Variation of the drag coefficient with time for flow past an impulsively started circular cylinder obtained from equation (1).

One method is to integrate the normal derivative of vorticity over a body:

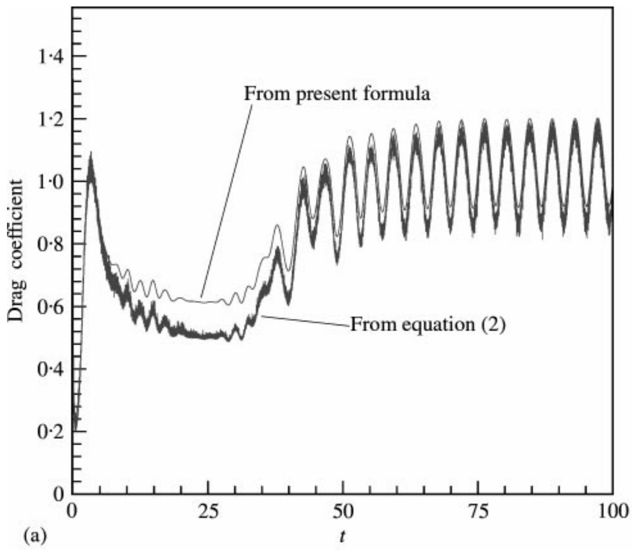
$$p(s) - p(0) = \int_0^s \mu \frac{\partial \omega}{\partial n} \, \mathrm{d}s, \tag{1}$$

where p and ω are the pressure and vorticity, respectively, $\partial/\partial n$ is the outward normal gradient on the body (see Figure1) and μ the dynamic viscosity.

The second method is to integrate the N–S equations from the body surface to infinity

$$p - p_\infty = \rho \int_{R_B}^{\infty} \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} + \nu \nabla \times (\omega \mathbf{e}_3) \right) \cdot \mathrm{d}\mathbf{r}, \tag{2}$$

where $\mathbf{e}_3$ is the unit vector perpendicular to flow plane, ∇ the gradient operator, $\mathbf{V}$ the velocity vector, and ρ and ν are the density and kinematic viscosity of fluid, respectively.



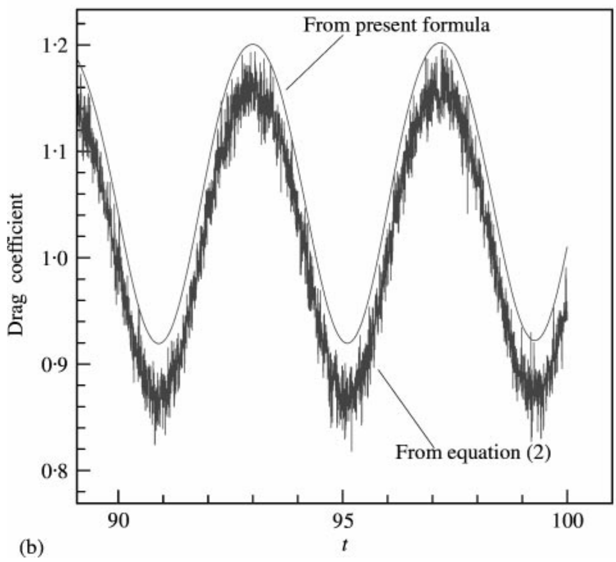


Figure 3. Variation of the drag coefficients with time for flow past an impulsively started circular cylinder obtained from equation (2) and equation (29): (a) $t = 0$–100; (b) $t = 89$–100.

The third method is to solve the Poisson equation of pressure obtained by taking divergence of the N–S equations

$$\nabla^2 p = -\rho \mathbf{\nabla} \cdot (\mathbf{V} \cdot \mathbf{\nabla} \mathbf{V}). \tag{3}$$

Among the three methods represented by equations (1–3), the first method has the obvious advantage of simplicity, if the flow field obtained is analytical or semi-analytical. This method was used by Collins & Dennis (1973) to calculate the drag and lift forces on a circular cylinder. However, if the flow field is not obtained analytically or semi-analytically, there will inevitably be errors in the numerical solutions of velocity and vorticity fields. These errors may be amplified and can lead to high-frequency numerical instability (see Figure 2) arising from the spatial derivative operation on the vorticity when the drag and lift forces are obtained from equation (1).

Instead of equation (1), Chang & Chern (1991*a*, *b*) utilized equation (2) to calculate the force acting on a circular cylinder. Our computational experiences (Chew *et al.* 1995) indicated that the force obtained from equation (2) is more accurate than that obtained from equation (1) for flow past a circular cylinder, but the high-frequency numerical instability (see Figure 3) contained in the force does not disappear completely. It is found that the numerical error basically originates from the differential operation of the term $\partial\mathbf{V}/\partial t$ in equation (2), which tends to magnify the error contained in the numerical solution of velocity field.

In the method based on equation (3), it is necessary to solve the Poisson equation for each time step or to solve a corresponding integral equation using boundary element method (Wang 1986). This would increase the CPU time significantly. Moreover, it is difficult to apply pressure boundary conditions on the body directly. Therefore, there is a need to develop an alternative method for calculating forces exerted on a body immersed in incompressible flows numerically.

Quartapelle & Napolitano (1983) proposed a method to evaluate the force acting on a body. They first introduced a harmonic vector function that not only is square summable in the flow region, but also satisfies the boundary conditions on the body surface and at far field. This vector function was then used to take a Hilbert scalar product with the N–S equations. By means of Gauss's theorem, the summation of the pressure force exerted by the fluid on the body was obtained. After adding the shear stress, the total force formula was derived. The key to this method is to find the vector function satisfying the boundary conditions as stated earlier. As an example, Quartapelle & Napolitano (1983) gave a harmonic vector function for the case of flow around a sphere. Later, Chang & Chern (1991*b*) found the vector function for flow around a circular cylinder.

As it is difficult to find a harmonic vector function for a body of complex geometry, Quartapelle & Napolitano (1983) did not simplify their formula to account for the effect of ambient flow acceleration (inertial force) and separate the contributing factors to the force. For engineering application, especially for ocean engineering, the inertial force is often a very important component. In this paper, an integral force formula including the inertial force is first presented. A general form of the harmonic vector function is then found for a 2-D body of arbitrary shape. Using the vector function obtained, a simplified force formula is derived. This formula describes explicitly and separately the effects of various physical factors contributing to the force. In addition, the formula can also be used for calculating the added mass and inertia coefficients of a body in inviscid flow.

2. BASIC INTEGRAL EQUATION FOR FORCE

Consider an incompressible viscous flow governed by the time-dependent N–S equations and the continuity equation

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V} = -\frac{\nabla p}{\rho} - \nu\nabla\times\boldsymbol{\omega}, \tag{4}$$

$$\nabla\cdot\mathbf{V} = 0. \tag{5}$$

Equations (4) and (5) are supplemented by the initial and boundary conditions:

$$\mathbf{V}|_{t=0} = \mathbf{V}_0, \tag{6}$$

$$\mathbf{V}|_{\Sigma_B} = \mathbf{V}_B(t), \quad \mathbf{V}|_{r\to\infty} = \mathbf{V}_\infty(t). \tag{7}$$

Let $\boldsymbol{\lambda}$ be the harmonic vector function satisfying

$$\nabla^2 \boldsymbol{\lambda} = \mathbf{0} \tag{8}$$

and taking the Hilbert scalar product of N–S equation by $\nabla\boldsymbol{\lambda}$, we have

$$\begin{aligned}\oint_{\Sigma_B} (\mathbf{n}\cdot\nabla\boldsymbol{\lambda})\,p\,\mathrm{d}s &= \rho\oint_{\Sigma_\infty}\left\{\mathbf{n}\cdot\left[\nabla\boldsymbol{\lambda}\left(\frac{p}{\rho}+\frac{V^2}{2}\right)+\frac{\partial\mathbf{V}}{\partial t}\boldsymbol{\lambda}\right]+\nu(\mathbf{n}\times\boldsymbol{\omega})\cdot\nabla\boldsymbol{\lambda}\right\}\mathrm{d}s\\ &\quad -\rho\int_{\Sigma_B}\left\{\nu(\mathbf{n}\times\boldsymbol{\omega})\cdot\nabla\boldsymbol{\lambda}+(\mathbf{n}\cdot\nabla\boldsymbol{\lambda})\frac{V^2}{2}+\mathbf{n}\cdot\frac{\partial\mathbf{V}}{\partial t}\boldsymbol{\lambda}\right\}\mathrm{d}s\\ &\quad +\rho\int_{\Omega}(\boldsymbol{\omega}\times\mathbf{V})\cdot\nabla\boldsymbol{\lambda}\,\mathrm{d}\Omega,\end{aligned} \tag{9}$$

where $\mathbf{n}$ is the outward-normal unit vector in the integration surface (see Figure 1) and Ω denotes the flow domain. It is obvious that if the harmonic vector function $\boldsymbol{\lambda}$ satisfies

$$\mathbf{n}\cdot\nabla\boldsymbol{\lambda}|_{\Sigma_B} = -\,\mathbf{n};\qquad \boldsymbol{\lambda}|_{r\to\infty} = \mathcal{O}\left(\frac{1}{r}\right), \tag{10}$$

the right-hand side of equation (9) becomes the contribution of the surface pressure to the force. After taking shear stress on the surface of the body into account, the total force acting on a body can be expressed as

$$\begin{aligned}\mathbf{F} &= -\oint_{\Sigma_B}(\mathbf{n}p+\mu\mathbf{n}\times\boldsymbol{\omega})\,\mathrm{d}s\\ &= \rho\oint_{\Sigma_\infty}\left\{\mathbf{n}\cdot\left[\nabla\boldsymbol{\lambda}\left(\frac{p}{\rho}+\frac{V^2}{2}\right)+\frac{\partial\mathbf{V}}{\partial t}\boldsymbol{\lambda}\right]+\nu(\mathbf{n}\times\boldsymbol{\omega})\cdot\nabla\boldsymbol{\lambda}\right\}\mathrm{d}s\\ &\quad -\rho\int_{\Sigma_B}\left\{\nu(\mathbf{n}\times\boldsymbol{\omega})\cdot\nabla\boldsymbol{\lambda}-\mathbf{n}\frac{V^2}{2}+\mathbf{n}\cdot\frac{\partial\mathbf{V}}{\partial t}\boldsymbol{\lambda}+\nu\mathbf{n}\times\boldsymbol{\omega}\right\}\mathrm{d}s\\ &\quad +\rho\int_{\Omega}(\boldsymbol{\omega}\times\mathbf{V})\cdot\nabla\boldsymbol{\lambda}\,\mathrm{d}\Omega.\end{aligned} \tag{11}$$

Equation (11) is a basic integral form of the force exerted by fluid on a body. It can be seen from this equation that the key to calculating the force is to obtain the harmonic vector function $\boldsymbol{\lambda}$ satisfying equation (8) and boundary conditions (10).

It may be noted that equation (11) and the second condition in equation (10) are different from that given by Quartapelle & Napolitano (1983). They considered only the force caused by body acceleration in a stationary fluid, while equation (11) includes the effect of not only body acceleration but also the acceleration of ambient flow. In many engineering applications, the inertial force caused by the acceleration of ambient flow is a very important factor.

3. A GENERAL FORM OF THE HARMONIC VECTOR FUNCTION

As mentioned earlier, the key to solving equation (11) is to obtain the harmonic vector function satisfying the boundary conditions. For 2-D flow, we shall prove that there exists a general form for the harmonic vector function $\boldsymbol{\lambda}$ satisfying equations (8) and (10) if the geometry of the body is simply connected.

It is well known from the complex function theory that for any simply connected 2-D body in the complex plane ($z = x + \mathrm{i}y$), there exists a conformal transformation

$$z = m\zeta + \sum_{k=1}^{\infty} \frac{m_k}{\zeta^k}, \qquad \zeta = \xi + \mathrm{i}\eta, \tag{12}$$

which maps the body into a unit circle in the ζ plane. We, therefore, state that *if the complex function* $\boldsymbol{\lambda}$ *is introduced by means of the coefficients in equation* (12)

$$\lambda = \frac{m}{\bar{\zeta}} - \sum_{k=1}^{\infty} \frac{m_k}{\zeta^k}, \tag{13}$$

where $\bar{\zeta}$ *denotes the conjugation of* ζ, *then the vector function* $\boldsymbol{\lambda}$ *constructed using the real and imaginary parts of complex function* λ *is the solution of equation* (8) *under the conditions given by equation* (10).

The proof of the above statement is as follows. Although the complex function defined by equation (13) is not analytical, it is obviously harmonic, i.e., its real and imaginary parts satisfy equation (8), respectively. It can be seen from equation (12) that $r = |z| = m\mathcal{O}(|\zeta|)|_{r\to\infty}$, and thus λ in equation (13) satisfies the second condition in equation (10). In order to verify the first condition in equation (10) being satisfied, operator $\mathbf{n}\cdot\boldsymbol{\nabla}$ is first expressed in the corresponding complex form using the method given by Milne-Thomson (1972), that is

$$\mathbf{n}\cdot\boldsymbol{\nabla} = N\frac{\partial}{\partial z} + \bar{N}\frac{\partial}{\partial \bar{z}}, \tag{14}$$

where $N = n_x + \mathrm{i}n_y$. By applying equation (14), the first condition in equation (10) can be rewritten as the following complex form:

$$\left(N\frac{\partial\lambda}{\partial z} + \bar{N}\frac{\partial\lambda}{\partial \bar{z}}\right)\bigg|_{\Sigma_B} = -N. \tag{15}$$

From equations (12) and (13), λ can be expressed as

$$\lambda = -z + m\left(\zeta + \frac{1}{\bar{\zeta}}\right).$$

By substituting λ into equation (15), one obtains

$$N\frac{\partial\lambda}{\partial z} + \bar{N}\frac{\partial\lambda}{\partial \bar{z}} = -N + m\left(N\frac{\partial}{\partial z} + \bar{N}\frac{\partial}{\partial \bar{z}}\right)\left(\zeta + \frac{1}{\bar{\zeta}}\right). \tag{16}$$

Owing to $N = \mathrm{d}z/\mathrm{i}|\mathrm{d}z|$ and $\mathrm{d}z\,(\partial/\partial z) = \mathrm{d}\zeta\,(\partial/\partial\zeta)$ on the boundary, we therefore have

$$\begin{aligned}\left(N\frac{\partial}{\partial z} + \bar{N}\frac{\partial}{\partial \bar{z}}\right)\left(\zeta + \frac{1}{\bar{\zeta}}\right) &= \frac{1}{\mathrm{i}|\mathrm{d}z|}\left(\mathrm{d}z\frac{\partial}{\partial z} - \mathrm{d}\bar{z}\frac{\partial}{\partial \bar{z}}\right)\left(\zeta + \frac{1}{\bar{\zeta}}\right)\\ &= \frac{1}{\mathrm{i}|\mathrm{d}z|}\left(\mathrm{d}\zeta\frac{\partial}{\partial \zeta} - \mathrm{d}\bar{\zeta}\frac{\partial}{\partial \bar{\zeta}}\right)\left(\zeta + \frac{1}{\bar{\zeta}}\right)\\ &= \frac{1}{\mathrm{i}|\mathrm{d}z|}\left(\mathrm{d}\zeta + \frac{\mathrm{d}\bar{\zeta}}{\bar{\zeta}^2}\right).\end{aligned} \tag{17}$$

As the body boundary is mapped into the unit circle in the ζ plane, on this unit circle there exist the following relationships:

$$\bar{\zeta} = \frac{1}{\zeta}, \qquad \mathrm{d}\bar{\zeta} = -\frac{1}{\zeta^2}\mathrm{d}\zeta. \tag{18}$$

Substituting equation (18) into equation (17), one obtains

$$\left(N\frac{\partial}{\partial z} + \bar{N}\frac{\partial}{\partial \bar{z}}\right)\left(\zeta + \frac{1}{\bar{\zeta}}\right) = 0. \tag{19}$$

Therefore, equation (16) reduces to equation (15) and the proof ends.

4. SIMPLIFIED FORCE FORMULA

It can be seen from equation (11) that even if $\boldsymbol{\lambda}$ or (λ as a complex number) is found, the formula is still too complicated for general application. In the following, equation (11) will be simplified after taking into consideration the physical features of flow field. It can be assumed according to equation (7) that in the region sufficiently far away from the body, vorticity is zero ($\omega = 0$) and the flow is potential; hence,

$$\oint_{\Sigma_\infty} (\mathbf{n} \times \boldsymbol{\omega}) \cdot \nabla\boldsymbol{\lambda}\, \mathrm{d}s = \mathbf{0}. \tag{20}$$

In this irrotational flow region, there exists the following Bernoulli's integral:

$$\frac{p}{\rho} + \frac{\mathbf{V}\cdot\mathbf{V}}{2} + \frac{\partial \phi_d}{\partial t} + \frac{\mathrm{d}\mathbf{V}_\infty(t)}{\mathrm{d}t}\cdot\mathbf{r} = c(t), \tag{21}$$

where $c(t)$ is a function of time and ϕ_d is disturbing velocity potential caused by the body and whole vorticity field. After the substitution of complex λ for vector $\boldsymbol{\lambda}$, the first integral in equation (11) becomes

$$\int_{\Sigma_\infty} \mathbf{n}\cdot\left(\left(\frac{p}{\rho} + \frac{V^2}{2}\right)\nabla\lambda + \frac{\partial \mathbf{V}}{\partial t}\lambda\right)\mathrm{d}s = \int_{\Sigma_\infty} \mathbf{n}\cdot\left(\left[c(t) - \frac{\partial\phi_d}{\partial t} - \frac{\mathrm{d}\mathbf{V}_\infty(t)}{\mathrm{d}t}\cdot\mathbf{r}\right]\nabla\lambda + \frac{\partial \mathbf{V}}{\partial t}\lambda\right)\mathrm{d}s.$$

According to the physics of the flow, although there are positive and negative vorticities in the fluid, the total vorticity should be zero if the body moves in the fluid translationally, and thus $\partial\phi_d/\partial t|_{r\to\infty} = \mathcal{O}(1/r)$. Similarly, the velocity at any point in the flow field can be divided into disturbing velocity $\mathbf{V}_\mathrm{d}$ plus $\mathbf{V}_\infty$, and $\mathbf{V}_d|_{r\to\infty} = \mathcal{O}(1/r^2)$. Therefore, the above equation reduces to

$$\int_{\Sigma_\infty} \mathbf{n}\cdot\left(\left(\frac{p}{\rho} + \frac{V^2}{2}\right)\nabla\lambda + \frac{\partial \mathbf{V}}{\partial t}\lambda\right)\mathrm{d}s = \int_{\Sigma_\infty} \mathbf{n}\cdot\left(\left(-\frac{\mathrm{d}\mathbf{V}_\infty}{\mathrm{d}t}\cdot\mathbf{r}\right)\nabla\lambda + \frac{\mathrm{d}\mathbf{V}_\infty}{\mathrm{d}t}\lambda\right)\mathrm{d}s. \tag{22}$$

Using the Gauss formula, we have

$$\begin{aligned}\int_{\Sigma_\infty} \mathbf{n}\cdot\nabla\lambda\left(-\frac{\mathrm{d}\mathbf{V}_\infty}{\mathrm{d}t}\cdot\mathbf{r}\right)\mathrm{d}s &= -\int_{\Sigma_B} \mathbf{n}\cdot\nabla\lambda\left(\frac{\mathrm{d}\mathbf{V}_\infty}{\mathrm{d}t}\cdot\mathbf{r}\right)\mathrm{d}s - \int_{\Omega} \nabla\lambda\cdot\frac{\mathrm{d}\mathbf{V}_\infty}{\mathrm{d}t}\,\mathrm{d}\Omega \\ &= \int_{\Sigma_B} N\frac{\mathrm{d}\mathbf{V}_\infty}{\mathrm{d}t}\cdot\mathbf{r}\,\mathrm{d}s + \int_{\Sigma_B} \lambda\mathbf{n}\cdot\frac{\mathrm{d}\mathbf{V}_\infty}{\mathrm{d}t}\,\mathrm{d}s - \int_{\Sigma_\infty} \lambda\mathbf{n}\cdot\frac{\mathrm{d}\mathbf{V}_\infty}{\mathrm{d}t}\,\mathrm{d}s,\end{aligned}$$

and hence

$$\int_{\Sigma_\infty} \mathbf{n}\cdot\left(\left(\frac{p}{\rho}+\frac{V^2}{2}\right)\nabla\lambda+\frac{\partial \mathbf{V}}{\partial t}\lambda\right)\mathrm{d}s=\int_{\Sigma_B}\left(N\frac{\mathrm{d}\mathbf{V}_\infty}{\mathrm{d}t}\cdot\mathbf{r}+\mathbf{n}\cdot\frac{\mathrm{d}\mathbf{V}_\infty}{\mathrm{d}t}\lambda\right)\mathrm{d}s. \tag{23}$$

If the surface shear stress $[-(\mathbf{n}\times\boldsymbol{\omega})\,\mathrm{d}s]$ is expressed by complex number $(\omega\,\mathrm{d}z)$ and $F=F_x+\mathrm{i}F_y$ is used to denote the force acting on the body, then equation (11) becomes

$$\begin{aligned}F=\rho\int_{\Sigma_B}&\left(N\frac{\mathrm{d}\mathbf{V}_\infty}{\mathrm{d}t}\cdot\mathbf{r}+\lambda\mathbf{n}\cdot\left(\frac{\mathrm{d}\mathbf{V}_\infty}{\mathrm{d}t}-\frac{\mathrm{d}\mathbf{V}_B}{\mathrm{d}t}\right)+N\frac{|\mathbf{V}_B|^2}{2}\right)\mathrm{d}s\\&-\mu\int_{\Sigma_B}(\mathbf{n}\times\boldsymbol{\omega})\cdot\nabla\lambda\,\mathrm{d}s+\mu\int_{\Sigma_B}\boldsymbol{\omega}\,\mathrm{d}z+\rho\int_{\Omega}(\boldsymbol{\omega}\times\mathbf{V})\cdot\nabla\lambda\,\mathrm{d}\Omega.\end{aligned} \tag{24}$$

To simplify equation (24) further, $V_\infty=u_{\infty x}+\mathrm{i}u_{\infty y}$ is used to replace $\mathbf{V}_\infty$, and $\forall$ is used to denote the area of the body. We then have

$$\int_{\Sigma_B} N\frac{\mathrm{d}\mathbf{V}_\infty}{\mathrm{d}t}\cdot\mathbf{r}\,\mathrm{d}s=\forall\frac{\mathrm{d}V_\infty}{\mathrm{d}t}, \tag{25}$$

$$\int_{\Sigma_B}\mathbf{n}\cdot\frac{\mathrm{d}\mathbf{V}_\infty}{\mathrm{d}t}\lambda\,\mathrm{d}s=\frac{1}{2\mathrm{i}}\int_{\Sigma_B}\lambda\left(\frac{\mathrm{d}\bar{V}_\infty}{\mathrm{d}t}\mathrm{d}z-\frac{\mathrm{d}V_\infty}{\mathrm{d}t}\mathrm{d}\bar{z}\right),$$

while

$$\begin{aligned}\frac{1}{2\mathrm{i}}\int_{\Sigma_B}\lambda\,\mathrm{d}z&=\frac{1}{2\mathrm{i}}\int_{\Sigma_B}\left[m\left(\zeta+\frac{1}{\bar{\zeta}}\right)-z\right]\mathrm{d}z\\&=\frac{m}{\mathrm{i}}\int_{\Sigma_B}\zeta\left(m-\frac{m_1}{\zeta^2}-2\frac{m_2}{\zeta^3}-3\frac{m_3}{\zeta^4}-\lambda\right)\mathrm{d}\zeta\\&=-2\pi m m_1\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2\mathrm{i}}\int_{\Sigma_B}\lambda\,\mathrm{d}\bar{z}&=\frac{1}{2\mathrm{i}}\int_{\Sigma_B}\left[m\left(\zeta+\frac{1}{\bar{\zeta}}\right)-z\right]\mathrm{d}\bar{z}\\&=\forall+\frac{m}{\mathrm{i}}\int_{\Sigma_B}\zeta\left(\bar{m}-\frac{\bar{m}_1}{\bar{\zeta}^2}-2\frac{\bar{m}_2}{\bar{\zeta}^3}-3\frac{\bar{m}_3}{\bar{\zeta}^4}-\lambda\right)\mathrm{d}\bar{\zeta}\\&=\forall-2\pi m\bar{m};\end{aligned}$$

therefore,

$$\int_{\Sigma_B}\mathbf{n}\cdot\frac{\mathrm{d}\mathbf{V}_\infty}{\mathrm{d}t}\lambda\,\mathrm{d}s=-\forall\frac{\mathrm{d}V_\infty}{\mathrm{d}t}+2\pi m\bar{m}\left(\frac{\mathrm{d}V_\infty}{\mathrm{d}t}-\frac{m_1}{\bar{m}}\frac{\mathrm{d}\bar{V}_\infty}{\mathrm{d}t}\right). \tag{26}$$

Similarly, we can obtain

$$\int_{\Sigma_B}\mathbf{n}\cdot\frac{\mathrm{d}\mathbf{V}_B}{\mathrm{d}t}\lambda\,\mathrm{d}s=-\forall\frac{\mathrm{d}V_B}{\mathrm{d}t}+2\pi m\bar{m}\left(\frac{\mathrm{d}V_B}{\mathrm{d}t}-\frac{m_1}{\bar{m}}\frac{\mathrm{d}\bar{V}_B}{\mathrm{d}t}\right). \tag{27}$$

The second integral in equation (24) can also be simplified further. In fact,

$$
\begin{aligned}
-\int_{\Sigma_B} (\mathbf{n}\times\boldsymbol{\omega})\cdot\nabla\lambda]\,\mathrm{d}s &= \mathrm{i}\int_{\Sigma_B}\omega\left(N\frac{\partial\lambda}{\partial z}-\bar{N}\frac{\partial\lambda}{\partial\bar{z}}\right)\mathrm{d}s \\
&= \int_{\Sigma_B}\omega\left(\frac{\partial\lambda}{\partial z}\mathrm{d}z+\frac{\partial\lambda}{\partial\bar{z}}\mathrm{d}\bar{z}\right)=\int_{\Sigma_B}\omega\left[-\mathrm{d}z+m\left(\mathrm{d}\zeta-\frac{1}{\bar{\zeta}^2}\mathrm{d}\bar{\zeta}\right)\right] \\
&= \int_{\Sigma_B}\omega(-\,\mathrm{d}z+2m\,\mathrm{d}\zeta).
\end{aligned}
\tag{28}
$$

After substituting equations (25–28) into equation (24), we finally obtain the formula:

$$
\begin{aligned}
F = \rho\forall\frac{\mathrm{d}V_B}{\mathrm{d}t} &- 2\pi m\bar{m}\rho\left(\frac{\mathrm{d}V_B}{\mathrm{d}t}-\frac{m_1}{\bar{m}}\frac{\mathrm{d}\bar{V}_B}{\mathrm{d}t}\right)+2\pi m\bar{m}\rho\left(\frac{\mathrm{d}V_\infty}{\mathrm{d}t}-\frac{m_1}{\bar{m}}\frac{\mathrm{d}\bar{V}_\infty}{\mathrm{d}t}\right) \\
&+ 2\mu m\int_{\Sigma_B}\omega\,\mathrm{d}\zeta+\rho\int_\Omega(\boldsymbol{\omega}\times\mathbf{V})\cdot\nabla\lambda\,\mathrm{d}\Omega+\rho\int_{\Sigma_B}\frac{\mathbf{V_B}\cdot\mathbf{V_B}}{2\mathrm{i}}\mathrm{d}z,
\end{aligned}
\tag{29}
$$

or

$$
\begin{aligned}
F = \rho\forall\frac{\mathrm{d}(V_B-V_\infty)}{\mathrm{d}t} &- 2\pi m\bar{m}\rho\left[\frac{\mathrm{d}(V_B-V_\infty)}{\mathrm{d}t}-\frac{m_1}{\bar{m}}\frac{\mathrm{d}(\bar{V}_B-\bar{V}_\infty)}{\mathrm{d}t}\right]+\rho\forall\frac{\mathrm{d}V_\infty}{\mathrm{d}t} \\
&+ 2\mu m\int_{\Sigma_B}\omega\,\mathrm{d}\zeta+\rho\int_\Omega(\boldsymbol{\omega}\times\mathbf{V})\cdot\nabla\lambda\,\mathrm{d}\Omega+\rho\int_{\Sigma_B}\frac{\mathbf{V}_B\cdot\mathbf{V}_B}{2\mathrm{i}}\mathrm{d}z.
\end{aligned}
\tag{29$'$}
$$

Equation (29) or (29′) is a simplified force formula expressed in terms of complex numbers. The real and imaginary parts denote the x- and y-components of the force vector acting on a body. In comparison to equation (11), equation (29) is much simpler and more convenient to apply. In the next section, we will discuss the physical meanings of each term in this formula in detail.

5. DISCUSSION ON THE FORCE FORMULA

It can be observed that equation (29) retains the integral of the term $\frac{1}{2}(\mathbf{V}_B\cdot\mathbf{V}_B)$ as its last term. For a body moving translationally in a viscous flow, the value of this integral term is equal to zero, as $\mathbf{V}_B$ is equal to the velocity of the body, which is only a function of time and does not change along the surface. The purpose of retaining this term here is to show that equation (29) is still applicable for inviscid flows ($\mu=0$). For instance, when vortices are embedded in an inviscid flow to simulate some practical phenomenon, equation (29) can be used to calculate the force acting on a body by setting $\mu=0$. For this case, $\mathbf{V}_B$ is the slip velocity on the surface of the body, and the integral term containing $\frac{1}{2}(\mathbf{V}_B\cdot\mathbf{V}_B)$ is nonzero.

The first two terms on the right-hand side of equation (29′) represent the added mass force caused by the translational acceleration of a body in an inviscid flow, and the third term represents the inertial force caused by the acceleration of ambient flow. The fourth term ($2\mu\mathrm{m}\int_{\Sigma_B}\omega\,\mathrm{d}\zeta$) consists of the contribution from the shear and normal stresses on the surface resulting from the viscosity of fluid. Hence, this term is due to the deformation of the fluid in a viscous flow. The fifth term $\rho\int_\Omega(\boldsymbol{\omega}\times\mathbf{V})\cdot\nabla\lambda\,\mathrm{d}\Omega$ is due to a nonzero vorticity field in the flow. It originates from the nonlinear convection part of N–S equations, and thus represents the contribution of the convection of vorticity in the fluid to the total force. The last term $\rho\int_{\Sigma_B}(\mathbf{V}_B\cdot\mathbf{V}_B/2\mathrm{i})\,\mathrm{d}z$ also originates from the convection part of N–S equations but is restricted to flow velocity at the wall. Its significance was described earlier.

Chang & Chern (1991*b*) obtained the following drag and lift coefficient formulae for steady flow past a circular cylinder based on Quartapelle & Napolitano's (1983) method:

$$C_D = \int_{\Omega} (\boldsymbol{\omega} \times \mathbf{V}) \cdot \nabla(\cos\theta/r)\, \mathrm{d}\Omega + \frac{2}{\mathrm{Re}} \int_{\Sigma_B} (\mathbf{n} \times \boldsymbol{\omega}) \cdot [\nabla(\cos\theta/r) + \mathbf{e}_1]\, \mathrm{d}s, \tag{30}$$

$$C_L = \int_{\Omega} (\boldsymbol{\omega} \times \mathbf{V}) \cdot \nabla(\sin\theta/r)\, \mathrm{d}\Omega + \frac{2}{\mathrm{Re}} \int_{\Sigma_B} (\mathbf{n} \times \boldsymbol{\omega}) \cdot [\nabla(\sin\theta/r) + \mathbf{e}_2]\, \mathrm{d}s, \tag{31}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the unit vectors parallel and perpendicular to the ambient flow, respectively, and Re is the Reynolds number (defined as $\mathrm{Re} = 2a\rho U_\infty/\mu$). It can easily be verified that equations (30) and (31) are special cases of equation (29). In fact, after setting $V_B = 0$, $V_\infty = U_\infty$, $\lambda = a/\bar{z}$ and nondimensionalization with the radius a, density ρ and U_∞, the real and imaginary parts of equation (29) can be shown to reduce to equations (30) and (31), respectively. However, equations (30) and (31) are valid only for the case of steady flow past a circular cylinder, whilst equation (29) is valid for an arbitrary body immersed in any viscous or inviscid, steady or unsteady incompressible flows.

It can be seen that equation (29) has the advantage of excluding the temporal derivative of velocity and spatial derivative of vorticity inside the flow field, except for $\mathbf{V}_B$ and $\mathbf{V}_\infty$. However, $\mathbf{V}_B$ and $\mathbf{V}_\infty$ are defined exactly at the boundary and at infinity and their temporal derivatives will not contribute to numerical differentiation errors. Thus, the force obtained by using this formula has less numerical differentiation error (see Figure 3). It is well known that the solution of velocity or vorticity field obtained by any numerical method inevitably contains numerical error. If equation (1) is used to calculate the force acting on the body, the magnifying coefficient of the error is proportional to $1/(\Delta n)$, where Δn is the grid length in the normal direction. For example, supposing that $\delta\omega(i, j)$ is the numerical error of vorticity at the node (i, j), the numerical solution of vorticity can be written as $\tilde{\omega}(i, j) = \omega(i, j) + \delta\omega(i, j)$, where $\omega(i, j)$ is the exact value of vorticity at the node. If one-side difference with two-order accuracy is used to denote the derivative of vorticity in the normal direction (j) on the boundary, we have

$$\frac{\partial\tilde{\omega}(i,1)}{\partial n} = \frac{4\omega(i,2) - 3\omega(i,1) - \omega(i,3)}{\Delta n} + \frac{4\delta\omega(i,2) - 3\delta\omega(i,1) - \delta\omega(i,3)}{\Delta n} + \mathcal{O}((\Delta n)^2). \tag{32}$$

It is clear that the error $\delta\omega$ may be magnified by $1/(\Delta n)$ times. Similarly, if equation (2) is used, the magnifying coefficient is proportional to $1/\Delta t$, where Δt is the time step. The advantage of the present formula is that the error in velocity and vorticity fields obtained from the numerical solution will not be magnified since the formula excludes the temporal derivative of velocity and spatial derivative of vorticity.

To illustrate the advantage of the present formula, the drag exerted on a circular cylinder started impulsively in a stationary fluid is calculated according to equations (1), (2) and (29) using the same velocity and vorticity fields obtained numerically by the vortex method (Chew *et al.* 1995). The Reynolds number of the flow considered here is 1000. For the numerical computation of the flow around a circular cylinder, it is necessary to choose a suitable outer boundary, r_{out}, where the zero vorticity condition as described in equation (20) can be applied. Sa & Chang (1990) showed that accurate numerical results can be obtained when the outer computational boundary is taken as $r_{\mathrm{out}} = 81{\cdot}3a$, where a is the radius of cylinder, while Behr *et al.* (1991) showed that $r_{\mathrm{out}} = 50a$ is sufficient. The outer boundary used here is located at $r_{\mathrm{out}} = 100a$ which is much larger than the values adopted by Sa & Chang and Behr *et al.* The computational domain is divided into 256×512 cells in θ- and r-directions, respectively, and the time step Δt is taken as $0{\cdot}001a/U_\infty = 0{\cdot}2$. The effects of spatial and temporal resolution on the convergence of numerical solution have been studied in detail by Cheng *et al.* (1997) and the above mesh size and time step are found to be optimal. It should be noted that the accuracy of the velocity and vorticity fields has no bearings on the validity of the present formula which is derived based purely on mathematical considerations.

In the present calculations, second-order difference is used to compute the normal derivative of vorticity in equation (1) and the temporal derivative of velocity in equation (2). The results of the drag coefficient obtained using equation (1) are plotted in Figure 2 while those using equation (3) are plotted in Figure 3. The high-frequency numerical instability arising from the spatial derivative of vorticity can clearly be seen in Figure 2. Although Figure 3(a) indicates that the mean drag curve obtained by equation (2) is close to that by equation (29), the magnified view from $t = 89$ to 100 in Figure 3(b) reveals that significant high-frequency instability arising from the temporal derivative of velocity is still present. However, equation (29) is able to generate a smooth drag coefficient curve, as evident in Figure 3, since it does not suffer from the disadvantages of computing spatial and temporal derivatives of vorticity and velocity filed.

In addition to the above, another important contribution of equation (29) is its ability to provide an alternative method for calculating the added mass force when a 2-D body is accelerated translationally in a flow. For the elliptical cylinder with long axis $2a$ in the x-direction and short axis $2b$, we have $m = (a + b)/2$, $m_1 = (a - b)/2$ and $\forall = \pi ab$. The added mass force is obtained from equation (29′) as

$$F_{\mathrm{addm}} = \frac{\rho\pi}{2}\left[(a^2 + b^2)\frac{\mathrm{d}(V_B - V_\infty)}{\mathrm{d}t} - (a^2 - b^2)\frac{\mathrm{d}(\bar{V}_B - \bar{V}_\infty)}{\mathrm{d}t}\right]. \tag{33}$$

For the cases of a circle, ellipse and plate, the added mass given by Sarpkaya & Isaacson (1981) are, respectively, $\rho\pi a^2$, $(\rho\pi b^2, \rho\pi a^2)$ and $\rho\pi a^2$. These are in agreement with what is predicted by equation (33).

The transformation

$$z = \frac{a}{2}\left(\frac{c}{a} - \mathrm{e}^{-\mathrm{i}\beta} + \zeta + \frac{(c/a)^2}{c/a - \mathrm{e}^{-\mathrm{i}\beta} + \zeta}\right)$$

is able to map any Joukowsky aerofoil in the z-plane into a unit circle in the ζ-plane, where c is the half-chord length, a and β are the two parameters controlling the thickness and camber of an aerofoil. In this case, it can easily be shown that $m = a/2$, $m_1 = (c/a)^2/2$ and $\forall = \pi/4\,(a^2 - c^4/(2ac\cos\beta - c^2))$. For any 2-D cylinder of arbitrary shape, m, m_1 and $\forall$ can be determined through analytical or numerical methods, such as a Fourier transform method, fast Fourier transform algorithm or some other numerical methods. Having obtained the coefficients m and m_1 as well as the area $\forall$, the added mass force acting on the cylinder can easily be determined.

For a 2-D cylinder of arbitrary shape, the added mass force is obtained conventionally from the potential function of the corresponding problem and kinetic energy law (Sarpkaya & Isaacson 1981). Generally, to obtain this potential function, it is necessary to find each of the coefficients in equation (12). It can be observed from equation (29) that the present added mass force terms are only related to the cross-sectional area of the body and the first two coefficients of the conformal transformation, m and m_1. It is thus necessary neither to determine all other coefficients in the conformal transformation nor to find the potential flow solution. Therefore, in the case of a cylinder moving translationally in a fluid, the method presented in equation (29) is simpler than the traditional method of calculating the added mass force.

6. CONCLUDING REMARKS

A general formula for calculating the force acting on an arbitrary two-dimensional rigid body in a viscous/inviscid or steady/unsteady incompressible flow has been obtained. This formula consists of three parts: the added mass force (and/or inertial force) in inviscid flow, the force caused by the deformation of fluid due to viscosity and the force caused by the convection of fluid with nonzero circulation. It describes separately the effect of these contributing factors to the force and enables them to be studied numerically in order to understand how they contribute to the total unsteady force under different flow conditions. It will be used to investigate oscillatory flow past a circular cylinder at different Keulegan–Carpenter numbers and frequency parameters, and numerically derive the drag and inertia coefficients commonly used in the Morison equation. These will be presented later in a separate paper.

As the formula excludes the temporal derivative of velocity and the spatial derivative of vorticity in the flow field, the numerical errors arising from the numerical solution of velocity or vorticity field will not be magnified, resulting in the calculated force to be smoother and more accurate. The formula also presents an alternative method for determining the added mass force of an arbitrary two-dimensional body accelerating translationally in a fluid. For bodies of simple shape, such as circles, ellipses and plates, the added masses predicted using the present method are in agreement with those obtained by the conventional method. For bodies of complex shape, the present method is simple since it requires the calculation of only the first two coefficients of the conformal transformation and the cross-sectional area.

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